Mathematics Unit - 02

* Students are Advised to Solve the Questions of Exercises in the Same Sequence 'or' as Directed by the **Faculty Members.** *

Conceptual Notes for IIT-JEE/PET/Boards

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Key Concept

1. Numbers and their sets

(i) Natural Numbers : N = { 1, 2, 3, 4,... }

(ii) Whole Numbers : W = { 0, 1, 2, 3, 4,.... }

(iii) Integer Numbers : I or Z = { ...–3, –2, –1, 0, 1, 2, 3,...}

$Z^+ = \{1, 2, 3, ...\}$, $Z^- = \{-1, -2, -3, ...\}$

$Z_0 = \{\pm 1, \pm 2, \pm 3, ...\}$

(iv) Rational Numbers : $Q = \{ p/q \,;\, p, q \in Z,\ q \neq 0 \}$

(v) Irrational numbers : The numbers which are not rational or which cannot be written in the form of p/q, called irrational numbers

Ex- $\{\sqrt{2}, \sqrt{3}, 2^{1/3}, 5^{1/4}, \pi, e, ...\}$

2. Tricotomy Law

The real numbers are ordered in magnitude means. If x and y be two real numbers then there will be one and only one of the following relation will hold.

$x < y,\quad x = y,\quad x > y$

3. Interval

(i) Close interval $[a, b] = \{x, a \le x \le b\}$

(ii) Open interval (a, b) or $]a, b[= \{x, a < x < b\}$

(iii) Semi open or semi close interval

$[a,b[$ or $[a,b) = \{x;\, a \le x < b\}$

$]a,b]$ or $(a,b] = \{x;\, a < x \le b\}$

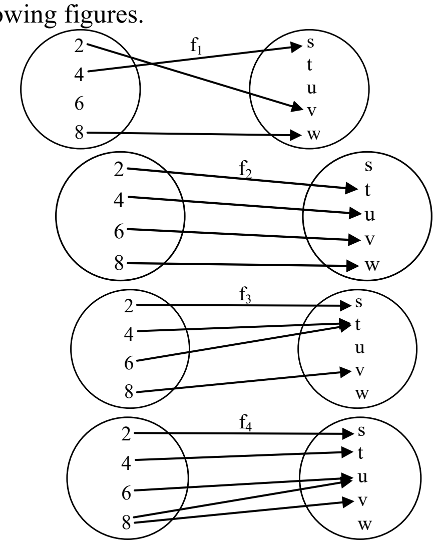
4. Function

Let A and B be two given sets and if each element $a \in A$ is associated with a unique element $b \in B$ under a rule f, then this relation is called Function.

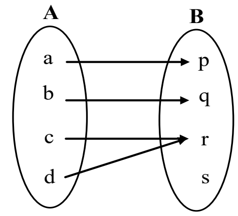
Here, b is called the image of a and a is called the pre-image of b under f.

➢**Example** :

Let A = {2, 4, 6, 8} and B = {s, t, u, v, w} be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures.



Now see that f_1 is not function from set A to set B, since there is an element 6 ∈ A which is not associated to any element of B. But f_2 and f_3 are the functions from A to B, because under f_2 and f_3 each element is A is associated to a unique element in B. But f_4 is not a function from A to B because an element 8∈A is associated to two element u & w in B.



Domain = {a, b, c, d} = A
Co-domain = {p, q, r, s} = B
Range = {p, q, r}

5. Domain and Range

Domain = All possible values of x for which f(x) exists.
Range = For all values of x, all possible values of f(x).

6. Domain and Range of some important functions

FUNCTION (y = f(x))	**DOMAIN** (i.e. values taken by x)	**RANGE** (i.e. values taken by f(x))
Algebraic Functions :		
(i) x^n, (n ∈ N)	R = (set of real numbers)	R, if n is odd $R^+ \cup \{0\}$, if n is even
(ii) $\frac{1}{x^n}$, (n ∈ N)	R – {0}	R – {0}, if n is odd R^+, if n is even
(iii) $x^{1/n}$, (n ∈ N)	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
(iv) $\frac{1}{x^{1/n}}$, (n ∈ N)	R– {0}, if n is odd R^+, if n is even	R – {0}, if n is od R^+, if n is even
Trigonometric Functions		
(i) sin x	R	[–1, 1]
(ii) cos x	R	[–1, 1]
(iii) tan x	R– (2k +1)π/2 , k ∈ I	R
(iv) sec x	$R - (2k+1)\frac{\pi}{2}$, k ∈ I	(–∞, –1] ∪ [1, ∞)
(v) cosec x	R – kπ, k ∈ I	(–∞, –1] ∪ [1, ∞)
(vi) cot x	R – kπ, k ∈ I	R
Inverse Circular Functions		
(i) $\sin^{-1} x$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $\cos^{-1} x$	[–1, 1]	[0, π]
(iii) $\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $\mathrm{cosec}^{-1} x$	(–∞, –1] ∪ [1, ∞)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(v) $\sec^{-1} x$	(–∞, –1] ∪ [1, ∞)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi) $\cot^{-1} x$	R	(0, π)
Exponential Functions		
(i) e^x	R	R^+
(ii) $e^{1/x}$	R – {0}	$R^+ - \{1\}$
(iii) a^x, a > 0	R	R^+
(iv) a1/x, a > 0	R– {0}	R+– {1}
Logarithmic Functions		
(i) logax, (a > 0) (a > > 1)	R+	R
(ii) logxa = $\frac{1}{\log_a x}$ (a > 0)	R+ – {1}	R – {0}
Integral part Functions		
(i) [x]	R	I
(ii) $\frac{1}{[x]}$	R – [0, 1)	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$

Fractional part Functions		
(i) {x}	R	[0, 1)
(ii) $\frac{1}{\{x\}}$	R – I	(1, ∞)
Modulus Functions		
(i) |x|	R	$R^+ \cup \{0\}$
(ii) $\frac{1}{\|x\|}$	R – {0}	R^+
Signum Function		
sgn (x) = $\frac{\|x\|}{x}$, $x \neq 0$ = 0, x = 0	R	{–1, 0, 1}
Constant Function		
say f(x) = c	R	{c}

7. Rules of domain

Dom (f + g + h.......) = Dom f ∩ Dom g ∩ Dom h ....
Dom (f – g) = Dom f ∩ Dom g
Dom (f × g × h ......) = Dom f ∩ Dom g ∩ Dom h ....
Dom (f / g) = Dom f ∩ Dom g – {x : g (x) = 0}

8. Kinds of Mapping

(i) One-one Function or Injection: A function f : A → B is said to be one-one if different elements of A have different images in B.

(ii) Many-one Function : A function f: A → B is called many-one, if two or more different elements of A have the same f-image in B.

(iii) Onto Function or Surjection: A function f : A → B is onto if the each element of B has its pre-image in A.
In other words, range of f = Co-domain of f

(iv) Into Function: A function f : A → B is into if there exist at least one element in B which is not the f-image of any element in A.
In other words, range of f ≠ co-domain of f

(v) One-one onto Function or bisection : A function f is said to be one-one onto if f is one-one and onto both.

(vi) One-one into Function: A function is said to be one-one into if f is one-one but not onto.

(vii) Many one-onto Function: A function f : A→B is many one-onto if f is onto but not one-one.
(i) $f : R \to R^+ \cup \{0\}$, $f(x) = x^2$.
(ii) f : R → [0, ∞), f(x) = |x|

(viii) Many one-into Function: A function f is said to be many one-into if it is neither one-one nor onto.
(i) f : R → R, f (x) = sin x
(ii) f : R → R, f(x) = |x|

(ix) Identity Function: Let A be any set and the function f : A → A be defined as f(x) = x, ∀ x ∈ A i.e. if each element of A is mapped by itself then f is called the identity function. It is represented by I_A. If A = {x, y, z} then I_A = {(x, x),(y, y), (z, z}

9. Even and Odd function

➢Even function
If we put (–x) in place of x in the given function and if f(–x) = f(x), ∀ x ∈ domain then function f(x) is called even function.

➢Odd function
If we put (–x) in place of x in the given function and if f(–x) = – f(x), ∀ x ∈ domain then f(x) is called odd function.

➢Properties of even and odd Function
(i) The product of two even fun. is even function.
(ii) The sum & diff. of two even fun. is even fun.
(iii) The sum & diff. of two odd fun. is odd function.
(iv) The product of two odd fun. is even function.
(v) The product of an even & an odd fun. is odd fun.
(vi) The sum of even & odd function is neither even nor odd function.

10. Explicit and Implicit Function

➢Explicit Function
A function is said to be explicit if it can be expressed directly in terms of the independent variable. y = f(x) or x = φ(y)

➢Implicit Function
A function is said to be implicit if it cannot be expressed directly in terms of the independent variable. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

11. Increasing and Decreasing Function

➢Increasing Function
A function f(x) is called increasing function in the domain D if the value of the function does not decrease by increasing the value of x. If $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ or $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or f ' (x) > 0 for increasing and f ' (x) ≥ 0 for not decreasing.

➢Decreasing Function
A function f (x) is said to be decreasing function in the domain D if the value of the function does not increase by increasing the value of x (variable).
if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ or $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or f ' (x) > 0 for decreasing and f ' (x) ≤ 0 for not decreasing.

12. Greatest Integer Function

A function is said to be greatest integer function if it is of the form of f(x) = [x] where [x] = integer equal or less than x. **i.e.** [4.2] = 4, [– 4. 4] = – 5

➢Properties of G.I.F :
(i) [x] = x if x is integer

(ii) $[x + I] = [x] + I$, if I is an integer

(iii) $[x + y] \geq [x] + [y]$

(iv) If $[\phi(x)] \geq I$ then $\phi(x) \geq I$

(v) If $[\phi(x)] \leq I$ then $\phi(x) < I + 1$

(vi) If $[x] > n \Rightarrow x \geq n + 1$

(vii) If $[x] < n \Rightarrow x < n, n \in I$

(viii) $[-x] = -[x]$ if $\forall x \in I$

(ix) $[-x] = -[x] - 1$ if $x \notin$ Integer

(x) $[x + y] = [x] + [y + x - [x]]\ \forall x, y \in R$

(xi) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+....+\left[x+\frac{n-1}{n}\right] = [nx]\ ; n \in N$

13. Fractional Part Function

It is denoted as $f(x) = \{x\}$ and defined as

(i) $\{x\} = f$ if $x = n + f$ where $n \in I$ and $0 \leq f < 1$

(ii) $\{x\} = x - [x]$

➢**Keep in mind** → For proper fraction $0 < f < 1$,

14. Signum Function

The signum function f is defined as

$$\text{Sgn } x = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases} = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

15. Modulus Function

It is given $n \in N$ by $y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

➢ **Properties of Modulus function:**

(i) $|x| \leq a \Rightarrow -a \leq x \leq a$

(ii) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$

(iii) $|x + y| = |x| + |y| \Rightarrow x, y \geq 0$ or $x \leq 0, y \leq 0$

(iv) $|x - y| = |x| - |y| \Rightarrow x \geq 0$ & $|x| \geq |y|$ or $x \leq 0$ and $y \leq 0$ and $|x| \geq |y|$

(v) $|x \pm y| \leq |x| + |y|$

(vi) $|x \pm y| \geq |x| - |y|$

16. Periodic Function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function f(x) will be periodic if a positive real number T exist such that, $f(x + T) = f(x)$, $\forall x \in$ Domain

Here the least positive value of T is called the period of the function.

For example, sin x, cos x, tan x are periodic functions with period 2π, 2π & π respectively.

➢**Note:**

(i) If function f (x) has period T then

(a) f(nx) has period T/ n

(b) f(x/n) has period nT

(c) f(ax + b) has period $\frac{T}{|a|}$

(ii) If the period of f(x) and g(x) are same (T) then the period of af(x)+bg(x) will also be T.

(iii) If the period of f(x) is T_1 and g (x) has T_2, then the period of f(x) ± g(x) will be LCM of T_1 and T_2 provided it satisfies the definition of periodic function.

17. Inverse Function

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f : A \to B$

$f^{-1}: B \to A$, $\quad f^{-1}(b) = a \quad \Rightarrow f(a) = b$

➢**Note :** For the existence of inverse function, it should be one-one and onto.

➢**Properties of Inverse Function**

(i) The inverse of a bisection is unique.

(ii) If $f : A \to B$ is a bisection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B respectively. Note that the graphs of f and g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x′, y′) corresponding to $y = x^2$ ($x \geq 0$) changes to (y′, x′) corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

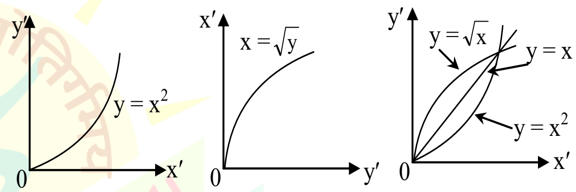


fig. 1 fig. 2 fig. 3

(iii) The inverse of a bijection is also a bijection.

(iv) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists & $(gof)^{-1} = f^{-1}og^{-1}$

18. Composite Function

If $f : A \to B$ and $g : B \to C$ are two function then the composite function of f and g, gof $A \to C$ will be defined as $gof(x) = g[f(x)]$, $\forall x \in A$.

➢**Properties of Composite Function**

(i) If f and g are two functions then for composite of two functions $fog \neq gof$.

(ii) Composite functions obeys the property of associativity i.e. fo (goh) = (fog) oh.

(iii) Composite function of two one-one onto functions if exist, will also be a one-one onto function.

➢**Algebra of function**

(i) (fog) (x) = f [g(x)]

(ii) (fof) (x) = f [f(x)]

(iii) (gog) (x) = g[g(x)]

(iv) (fg) (x) = f(x) . g(x)

(v) (f ± g) (x) = f(x) ± g(x)

(vi) $(f/g)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

* Composite functions in not commutative

* Let f and g are two functions then if f & g are injective or subjective or objective then "gof" also injective or subjective or objective.

19. Even and Odd function

➢Even Extension:

If a function f(x) is defined on the interval [0, a], $0 \le x \le a \Rightarrow -a \le -x \le 0$ we define f(x) in the [– a, 0] such that f(x) = f(–x).

Let $I(x) = \begin{cases} f(x) & : x \in [0,a] \\ f(-x) & : x \in [-a,0] \end{cases}$

➢Odd Extension:

If a function f(x) is defined on the interval [0, a], $0 \le x \le a \Rightarrow -a \le -x \le 0$

∴ $x \in [-a, 0]$, we define f(x) = – f(–x) Let I be the odd extension then $I(x) = \begin{cases} f(x), & x \in [0,a] \\ -f(-x), & x \in [-a,0] \end{cases}$

20. Some very important point

(a) If x, y are independent variables then :

(i) If f(x, y) = f(x) + f(y) then ⇒ f(x) = k logx

(ii) If f(x y) = f(x) . f(y) then ⇒ $f(x) = x^n$, $n \in R$

(iii) If f(x + y) = f(x).f(y) ⇒ $f(x) = a^{kx}$

(iv) If f(x + y) = f(x) + f(y) ⇒ f(x) = x

(v) if f(x + y) = f(x) = f(y) ⇒ f(x) = k, here k is constant

(vi) By considering a general n^{th} degree polynomial and writing the expression

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \quad \Rightarrow \quad f(x) = \pm x^n + 1$$

(b) Algebraic Functions

(i) **Polynomial function:** A function having the form $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\dots\dots a_nx^n$, where a_0, a_1, a_2 .... a_n are real constant, $a_n \neq 0$ and $n \in N$ called rational integral function or polynomial of degree n.

(ii) **Rational Function:** The ratio of two polynomial is called Fraction Rational function or simply rational function. e.g. $y = \dfrac{x^{12} + x^2 - 1}{x^6 + x^4 + 1}$

(iii) **Irrational Function:** Functions with operations of addition, subtraction, multiplication, division and raising to power with non-integral rational exponent are called irrational functions.

(I) $y = \sqrt{x}$ (II) $y = \dfrac{\sqrt{x^3+1} - \sqrt{x^{11}}}{\sqrt{x^2+x+1}}$

Such type of fun. are called Irrational function.

(iv) **Transcendental function :** All those function who has infinite terms while expanded are called transdental function. for example all trigonometrical function.

Inverse trigonometrically function, exponential function, logarithmic function etc.

e.g. f(x) = sin x, $y = \cos^{-1} x$

$y = \log_e x$, $y = \sqrt{\log_e x - \sin^{-1} x}$

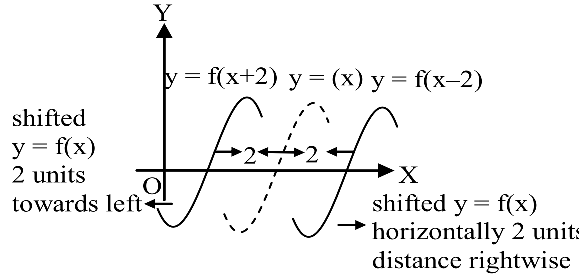
(c) Mapping :

One-one or injective mapping or monomorphism. If f : A → B is one-one mapping A has m element and B has n element hence the no. of mappings $= \begin{cases} {}^nP_m, & n \ge m \\ 0, & n < m \end{cases}$

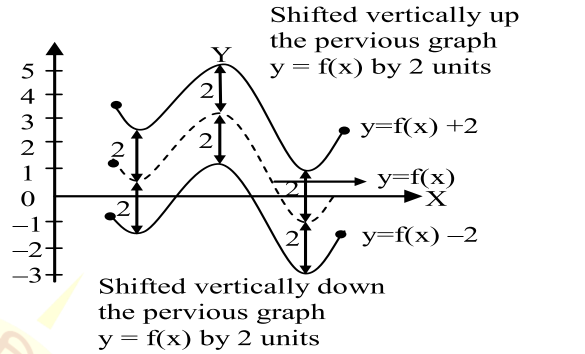
(d) If graph of y = f(x) be known then to find the graph of

(i) y = f(x – a) or y = f(x + a)

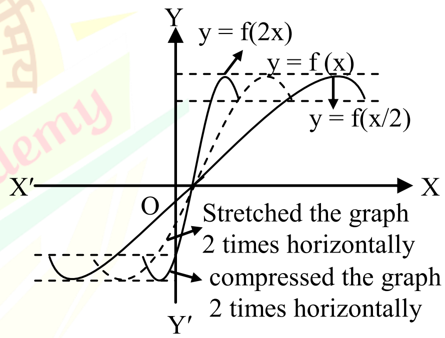
To find y = f(x – a) (Let a = 2)



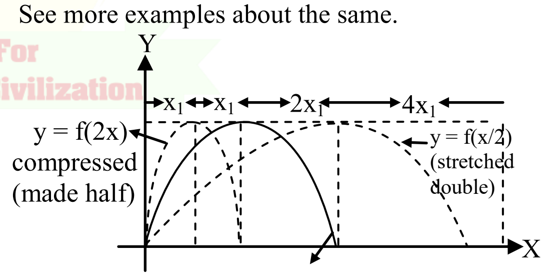
(ii) y = f (x) + a : or y = f (x) – a (Let a = 2)



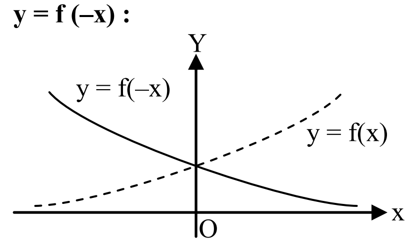
(iii) y = f(x/a) or y = f(ax) : (Let a = 2, 1/2)



See more examples about the same.



(iv) y = f (–x) :

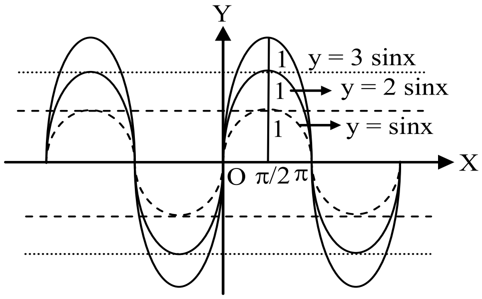


Reflection of y = f(x) w.r.t. axis of y is y = f(–x)

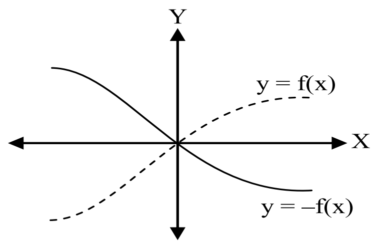
(v) To find y = k f (x) :

➢Rule – Stretch the previous graph k times vertically

e.g. see below y = 2 sin x, y = 3 sin x

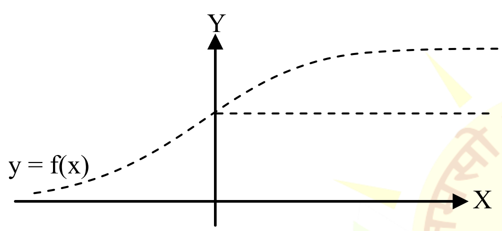


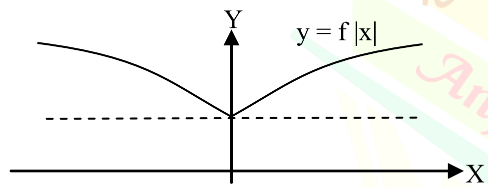
(vi) y = – f(x) :



Reflection of y = f(x) w.r.t. axis of x is y = –f(x)

(vii) To find y = f|x| :



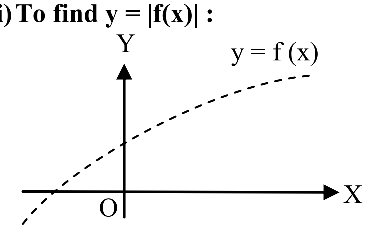


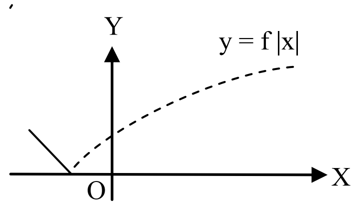
➢**RULE :** Neglect the graph lying in IInd and IIIrd quadrant and, Take the image of graph lying in I and IVth quadrant w. r. t. axis of y.

The original graph including its image is called y = f | x |.

Here we took the image of the portion lying in first quadrant about axis of y and left the portion which was lying in second quadrant.

(viii) To find y = |f(x)| :





➢**Rule :** Take the image of the portion line below axis of x about axis of x. Remain as it is the portion above the axis of x.

➢ Solved Examples

Ex.1 Let, $f(x) = x + 1, \quad x \le 1$

$\qquad = 2x + 1, \quad 1 < x \le 2$

$g(x) = x^2, \quad -1 \le x < 2$

$\qquad = x + 2, \quad 2 \le x \le 3$

Find fog and gof.

Sol. $f\{g(x)\} = g(x) + 1, \quad g(x) \le 1$

$\qquad = 2g(x) + 1, \quad 1 < g(x) \le 2$

$\Rightarrow f\{g(x)\} = x^2 + 1, \quad -1 \le x \le 1$

$\qquad = 2x^2 + 1, \quad 1 < x \le \sqrt{2}$

$g\{f(x)\} = \{f(x)\}^2, \quad -1 \le f(x) < 2$

$\qquad = f(x) + 2, \quad 2 \le f(x) \le 3$

$gof(x) = (x+1)^2, \quad -2 \le x < 1$

$\qquad = (x+1)^2 \quad -2 \le x \le 1$

Ex.2 Find the range of the following function:

$$f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$$

Sol. $\because \; f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

$= \log_2\left(\sin\left(\pi - \frac{\pi}{4}\right) + 3\right) \quad = \quad y \text{ (let)}$

$\Rightarrow 2^y = \sin\left(\pi - \frac{\pi}{4}\right) + 3 \quad \Rightarrow 2^y - 3 = \sin\left(\pi - \frac{\pi}{4}\right)$

But $-1 \le \sin\left(\pi - \frac{\pi}{4}\right) \le 1 \qquad \therefore \; -1 \le 2^y - 3 \le 1$

$\Rightarrow 2 \le 2^y \le 4 \qquad \Rightarrow 2^1 \le 2^y \le 2^2$

Hence, $y \in [1, 2]$. So, Range of f(x) is [1, 2].

Ex.3 Find the period of the following function

$f(x) = e^{x-[x] + |\cos \pi x| + |\cos 2\pi x| + ... + |\cos n\pi x|}$,

[.] is greatest integer function.

Sol. $f(x) = e^{x - [x] + |\cos \pi x| + |\cos 2\pi x| + ... + |\cos n\pi x|}$

Period of x – [x] = 1

Period of | cos πx | = 1

Period of | cos 2πx | = 1/2

....

....

Period of | cos nπx | = 1/n

So period of f(x) will be L.C.M. of all periods = 1.

Ex.4 Find the inverse of the following function :

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$$

Sol. Let $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Let $f(x) = y \qquad \therefore x = f^{-1}(y)$

$$\Rightarrow x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

$$\Rightarrow\ f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

$$\Rightarrow\ \therefore\ f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ x^2/64, & x > 16 \end{cases}$$

Ex.5 Let a function $f : R \to R$ be defined as $f(x) = x - [x]$, (where $[x]$ is a greatest integer $\le x$), for all $x \in R$, Is the function bijective ?

Sol. Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)\ \Rightarrow\ x_1 - [x_1] = x_2 - [x_2]$

$\Rightarrow x_1 \ne x_2\ \therefore$ The function is not bijective.

Ex.6 If $f(x) = \begin{cases} x^3+1, & x < 0 \\ x^2+1, & x \ge 0 \end{cases}$, $g(x) = \begin{cases} (x-1)^{1/3}, & x < 1 \\ (x-1)^{1/2}, & x \ge 1 \end{cases}$

Compute gof (x).

Sol. We have gof (x) = g (f (x))

$$= \begin{cases} g(x^3+1), & x < 0 \\ g(x^2+1), & x \ge 0 \end{cases} = \begin{cases} (x^3+1-1)^{1/3} & x < 0 \\ (x^2+1-1)^{1/2}, & x \ge 0 \end{cases}$$

$$= \begin{cases} x, & x < 0 \\ x, & x \ge 0 \end{cases} = x \text{ for all } x.$$

Hence, gof (x) = x for all x.

Ex.7 Let $f : R \to R$ be given by

$f(x) = (x+1)^2 - 1,\ x \ge -1$. Show that f is invertible.

Also, find the set $S = \{x : f(x) = f^{-1}(x)\}$.

Sol. In order to show that f (x) is invertible, it is sufficient to show that f (x) is a bijection.

f is an injection: For any $x, y \in R$ satisfying

$x \ge -1,\ y \ge -1$, We have $f(x) = f(y)$

$\Rightarrow\ (x+1)^2 - 1 = (y+1)^2 - 1$

$\Rightarrow\ x^2 + 2x = y^2 + 2y\ \Rightarrow\ x^2 - y^2 = -2(x-y)$

$\Rightarrow\ (x-y)(x+y) = -2(x-y)$

$\Rightarrow\ (x-y)[x+y+2] = 0\ \Rightarrow x - y = 0$ or $x + y + 2 = 0$

$\Rightarrow\ x = y$ or $x = y = -1$

Thus, $f(x) = f(y)\ \Rightarrow x = y$ for all $x \ge -1,\ y \ge -1$.

So, f (x) is an injection.

f is a surjection : For all $y \ge -1$ there exists.

$x = -1 + \sqrt{y+1} \ge -1$ such that $f(x) = y$

So, f (x) is a surjection.

Hence, f is a bijection. Consequently, it is invertible.

$f(x) = f^{-1}(x)\ \Rightarrow f(x) = x$

$(x+1)^2 - 1 = x\ \Rightarrow x = 0, -1$

Ex.8 Let $f(x) = x^2 + x$ be defined on the interval [0, 2]. Find the odd and even extensions of f (x) in the interval [–2, 2].

Sol. Odd extension.

$$f(x) = \begin{cases} f(x), & 0 \le x \le 2 \\ -f(-x), & -2 \le x < 0 \end{cases} = \begin{cases} x^2+x, & 0 \le x \le 2 \\ -x^2+x, & -2 \le x < 0 \end{cases}$$

Even extension

$$f(x) = \begin{cases} f(x), & 0 \le x \le 2 \\ f(-x), & -2 \le x < 0 \end{cases} = \begin{cases} x^2+x, & 0 \le x \le 2 \\ x^2-x, & -2 \le x < 0 \end{cases}$$

Ex.9 Find the domain and range of the function $f(x) = \sqrt{2-x} + \sqrt{1+x}$

Sol. Domain of $f(x) = \{x \mid 2 - x \ge 0 \text{ and } 1 + x \ge 0\}$

$\therefore$ domain of f(x) = [–1, 2]

Again, $\{f(x)\}^2 = \left(\sqrt{2-x} + \sqrt{1+x}\right)^2 = 3 + 2\sqrt{(2-x)(1+x)}$

$$= 3 + 2\sqrt{2+x-x^2} = 3 + 2\sqrt{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2}$$

$\therefore$ the greatest value of $\{f(x)\}^2 = 3 + 2.\sqrt{\frac{9}{4}} = 6$,

when $x = \frac{1}{2}$ the least value of $\{f(x)\}^2 = 3 + 0 = 3$,

when $x - \frac{1}{2} = \frac{3}{2}$, i.e. x = 2

$\therefore$ the greatest value of $f(x) = \sqrt{6}$

and the least value of $f(x) = \sqrt{3}$

$\therefore$ range of $f(x) = [\sqrt{3}, \sqrt{6}]$

Ex.10 Let $f(x) = \tan x,\ x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $g(x) = \sqrt{1-x^2}$. Determine fog and gof.

Sol. From the given domain of f $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ we conclude that its range $]-\infty, \infty[$ i.e. whole of R

Domain of g is $1 - x^2 \ge 0$ or $x^2 - 1 \le 0$

or $(x+1)(x-1) \le 0$ or $-1 \le x \le 1$ or [–1, 1]

and for range of g, $y = \sqrt{1-x^2}$

since $x^2 \le 1\ \ \therefore\ y \in [0, 1]$

$(fog)\,x = f(g(x)) = f\{\sqrt{1-x^2}\}$

$= f(t)$, where $t = \sqrt{1-x^2} \in [0, 1]$

range of $g \subset \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ which is domain of f.

$= \tan t = \tan \sqrt{1-x^2}$

$(gof)\,x = g(f(x)) = g(\tan x)$

$= g(t)$ where $t = \tan x \in$ range of f = R

But R is not a subset of domain of g = [–1, 1]

Hence gof is not defined.

Ex.11 The value of $n \in I$ for which the function $f(x) = \frac{\sin nx}{\sin(x/n)}$ has 4π as its period is -

(A) 2 (B) 3 (C) 5 (D) 4

Sol. For n = 2, we have $\frac{\sin 2x}{\sin(x/2)} = 4(\cos x/2)\cos x$.

The period of cosx is 2π, & that of cos(x/2) is 4π.

Hence the period of $\frac{\sin 2x}{\sin(x/2)}$ is 4π.

Also, the period of $\frac{\sin 3x}{\sin(x/3)}$, $\frac{\sin 5x}{\sin(x/5)}$ and $\frac{\sin 4x}{\sin(x/4)}$ cannot be 4π. **Ans.[A]**

Ex.12 Find the range of the following fun. $f(x) = \dfrac{3}{2-x^2}$

Sol. Let $y = \dfrac{3}{2-x^2} = f(x)$ ....(1)

The function y is not defined for $x = \pm\sqrt{2}$

From (1), $x^2 = \dfrac{2y-3}{y}$ since for real x, $x^2 \geq 0$,

We have $\dfrac{2y-3}{y} \geq 0$

$\therefore\ y \geq 3/2$ or $y < 0$ (Note that $y \neq 0$)

Hence the range of the function is

$]-\infty, 0[\ \cup [3/2, \infty)$

Ex.13 Prove that even functions do not have inverse.

Sol. Even functions are many one function and for the existence of inverse function should be one-one. Hence inverse of an even fun. will not exist.

Ex.14 Prove that periodic functions do not have inverse.

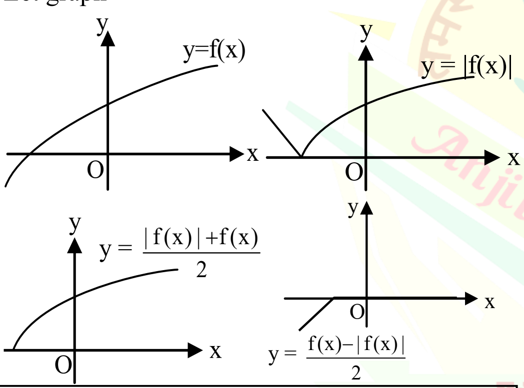
Sol. f (x) is periodic

$\Rightarrow$ f is many one $\Rightarrow$ f^{-1} does not exist.

Ex.15 Knowing the graph of y = f (x) draw

$y = \dfrac{f(x)+|f(x)|}{2}$ and $y = \dfrac{f(x)-|f(x)|}{2}$

Sol. Let graph



Problem for Home Practice

Question based on Domain

Q.1 Domain of $y = \log_{10}\left(\dfrac{5x-x^2}{4}\right)$:

(A) (0, 5) (B) [1, 4]
(C) $(-\infty, 0) \cup (5, \infty)$ (D) $(-\infty, 1) \cup (4, \infty)$

Q.2 The domain of definition of $f(x) = \dfrac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{x^2+2x+8}}$ is:

(A) (1, 4) (B) (–2, 4) (C) (2, 4) (D) $[2, \infty)$

Q.3 The function

$f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$ is defined on the set S, where S is equal to:

(A) {0, 3} (B) (0, 3) (C) {0, –3} (D) [–3, 0]

Q.4 The domain of $\sqrt{\sec^{-1}\left(\dfrac{2-|x|}{4}\right)}$ is

(A) R (B) R – (–1, 1)
(C) R – (–3, 3) (D) R – (– 6, 6)

Q.5 The domain of the function

$f(x) = {}^{24-x}C_{3x-1} + {}^{40-6x}C_{8x-10}$ is -

(A) {2, 3} (B) {1, 2, 3}
(C) {1, 2, 3, 4} (D) None of these

Question based on Range

Q.6 The range of the function $y = \dfrac{1}{2-\sin 3x}$ is :

(A) $\left(\dfrac{1}{3}, 1\right)$ (B) $\left[\dfrac{1}{3}, 1\right)$ (C) $\left[\dfrac{1}{3}, 1\right]$ (D) None

Q.7 The value of the function

$f(x) = \dfrac{x^2-3x+2}{x^2+x-6}$ lies in the interval -

(A) $(-\infty, \infty) - \left\{\dfrac{1}{5}, 1\right\}$ (B) $(-\infty, \infty)$
(C) $(-\infty, \infty) - \{1\}$ (D) None of these

Q.8 Find the range of the following function,

$y = \log_{\sqrt{7}}(\sqrt{2}(\sin x - \cos x) + 5)$

(A) R (B) Z
(C) $[\log_7 4, \log_7 5]$ (D) $[2 \log_7 3, 2]$

Q.9 Which of the following function (s) has the range [–1, 1]

(A) f(x) = cos (2 sin x) (B) $g(x) = \cos\left(1 - \dfrac{1}{1+x^2}\right)$
(C) $h(x) = \sin(\log_2 x)$ (D) $k(x) = \tan(e^x)$

Question based on Kinds of functions

Q.10 Let f : R $\rightarrow$ R be a function defined by

$f(x) = \dfrac{x^2+2x+5}{x^2+x+1}$ is :

(A) one-one and into (B) one-one and onto
(C) many-one and onto (D) many-one and into

Q.11 The function f : $[2, \infty) \rightarrow$ Y defined by $f(x) = x^2 - 4x + 5$ is both one–one & onto if:

(A) Y = R (B) $Y = [1, \infty)$
(C) $Y = [4, \infty)$ (D) $Y = [5, \infty)$

Q.12 Let f : R $\rightarrow$ R be a function defined by $f(x) = x^3 + x^2 + 3x + \sin x$. Then f is :

(A) one– one & onto (B) one –one & into
(C) many one & onto (D) many one & into

Q.13 Which of the following function from $A = \{x : -1 \leq x \leq 1\}$ to itself are bijections-

(A) f(x) = x/2 (B) g(x) = sin (πx/2)
(C) h(x) = |x| (D) $k(x) = x^2$

Question based on Inverse function

Q.14 If $f(x) = x^3 - 1$ and domain of f = {0,1, 2, 3}, then domain of f^{-1} is -

(A) {0, 1, 2, 3} (B) {1, 0, –7, –26}
(C) {–1, 0, 7, 26} (D) {0, –1,– 2, –3}

Q.15 The inverse of the function $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is

(A) $\dfrac{1}{2}\log\dfrac{1+x}{1-x}$ (B) $\dfrac{1}{2}\log\dfrac{2+x}{2-x}$
(C) $\dfrac{1}{2}\log\dfrac{1-x}{1+x}$ (D) 2 log (1+ x)

Question based on Composite function

Q.16 The function f(x) is defined in [0, 1] then the domain of definition of the function $f[\ell n\,(1-x^2)]$ is given by :
(A) $x \in \{0\}$ (B) $x \in [-\sqrt{1+e}-1] \cup [1 + \sqrt{1+e}\,]$
(C) $x \in (-\infty, \infty)$ (D) None of these

Question base don Periodic function

Q.17 If $f: R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = 2\ \forall\ x \in R$ then the period (may not be fundamental period) of f (x) is
(A) 3 (B) 4 (C) 7 (D) 6

Q.18 The fundamental period of the function:
$f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin 3\pi x + \cos 4\pi x + \ldots + \sin (2n-1)\pi x + \cos 2n\pi x$ for every $a, b \in R$ is:
(where [.] denotes the greatest integer function)
(A) 2 (B) 4 (C) 1 (D) 0

Q.19 Let $f(x) = \sin \sqrt{[a]}\, x$ (where [ ] denotes the greatest integers function). If f is periodic with fundamental period π, then a belongs to -
(A) [2, 3) (B) {4, 5} (C) [4, 5] (D) [4, 5)

Question based on Even and odd function

Q.20 Which of the following is an even function?
(A) $x\dfrac{a^x-1}{a^x+1}$ (B) tan x
(C) $\dfrac{a^x - a^{-x}}{2}$ (D) $\dfrac{a^x+1}{a^x-1}$

Q.21 Which of the following function is an odd function
(A) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$
(B) $f(x) = x\left(\dfrac{a^x+1}{a^x-1}\right)$
(C) $f(x) = \log\left(\dfrac{1-x}{1+x^2}\right)$ (D) f(x) = k (constant)

Question based on Miscellaneous

Q.22 The set of points for which $f(x) = \cos(\sin x) > 0$ contains -
(A) $(-\infty, 0]$ (B) [–1, 1]
(C) $(-\infty, \infty)$ (D) All are correct

Q.23 If [x] stands for the greatest integer function, then the value of
$$\left[\frac{1}{2}+\frac{1}{1000}\right]+\left[\frac{1}{2}+\frac{2}{1000}\right]+\ldots\ldots+\left[\frac{1}{2}+\frac{999}{1000}\right]$$
(A) 498 (B) 499 (C) 500 (D) 501

Q.24 Let the function $f(x) = 3x^2 - 4x + 8\log(1+|x|)$ be defined on the interval [0, 1]. The even extension of f(x) to the interval [–1, 0] is -
(A) $3x^2 + 4x + 8\log(1+|x|)$
(B) $3x^2 - 4x + 8\log(1+|x|)$
(C) $3x^2 + 4x - 8\log(1+|x|)$
(D) $3x^2 - 4x - 8\log(1+|x|)$

Q.25 Let $f: N \to N$ where $f(x) = x + (-1)^{x-1}$ then f is-
(A) Inverse of itself (B) even function
(C) periodic (D) identity

Old IIT-JEE Questions

Q.1 Let $f(x) = (1+b^2)x^2 + 2bx + 1$ and let m(b) be the min. value of f(x). As b varies, the range of m(b) is -
(A) [0, 1] (B) [0, 1/2] (C) [1/2, 1] (D) (0, 1]

Q.2 Let E = {1, 2, 3, 4} and F = {1, 2}. Then the number of onto functions from E to F is-
(A) 14 (B) 16 (C) 12 (D) 8

Q.3 Let $f(x) = \dfrac{\alpha x}{x+1}$, $x \neq -1$, then for what value of α, $f\{f(x)\} = x$.
(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 1 (D) –1

Q.4 The domain of definition of
$f(x) = \dfrac{\log_2(x+3)}{x^2+3x+2}$ is –
(A) R – {–2, +2} (B) (– 2, ∞)
(C) R– {–1, –2, –3} (D) (–3, ∞) / {–1, –2}

Q.5 If $f: [1, \infty) \to [2, \infty)$ is given by
$f(x) = x + \dfrac{1}{x}$ then $f^{-1}(x)$ equals -
(A) $\dfrac{x+\sqrt{x^2-4}}{2}$ (B) $\dfrac{x}{1+x^2}$
(C) $\dfrac{x-\sqrt{x^2-4}}{2}$ (D) $1 + \sqrt{x^2-4}$

Q.6 Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ Then for all x, f (g(x) is equal to-
(A) x (B) f (x) (C) 1 (D) g (x)

Q.7 Suppose $f(x) = (x+1)^2$ for $x \geq -1$. If g(x) is the function whose graph is the reflection of the graph of f(x) with respect to the line y = x, then g(x) equals–
(A) $-\sqrt{x} - 1,\ x \geq 0$ (B) $\dfrac{1}{(x+1)^2},\ x > -1$
(C) $\sqrt{x+1},\ x \geq -1$ (D) $\sqrt{x} - 1,\ x \geq 0$

Q.8 Let function $f: R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is–
(A) one to one and onto
(B) one to one but not onto
(C) onto but not one to one
(D) neither one to one nor onto

Q.9 Let $f(x) = \dfrac{x}{1+x}$ defined as $[0, \infty) \to [0, \infty)$, f(x) is–
(A) one- one & onto (B) one–one but not onto
(C) not one–one but onto (D) neither one–one nor onto

Q.10 If $f(x) = \sqrt{x^2+x} + \dfrac{\tan^2\alpha}{\sqrt{x^2+x}}$, $\alpha \in (0, \pi/2)$, x > 0 then value of f(x) is greater than or equal to–
(A) 2 (B) 2 tan α (C) 5/2 (D) sec α

Q.11 Find the range of $f(x) = \dfrac{x^2+x+2}{x^2+x+1}$ is –
(A) (1, ∞) (B) $\left(1, \dfrac{11}{7}\right]$ (C) $\left(1, \dfrac{7}{3}\right]$ (D) $\left(1, \dfrac{7}{5}\right]$

Q.12 Domain of $f(x)=\sqrt{\sin^{-1}(2x)+\pi/6}$ is–

(A) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (B) $\left[-\frac{1}{2},\frac{1}{2}\right]$

(C) $\left[-\frac{1}{4},\frac{1}{4}\right]$ (D) $\left[-\frac{1}{2},\frac{1}{4}\right]$

Q.13 Let $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$, then $g(f(x))$ will be invertible for the domain-

(A) $x \in [0, \pi]$ (B) $x \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

(C) $x \in \left[0,\frac{\pi}{2}\right]$ (D) $x \in \left[-\frac{\pi}{2},0\right)$

Q.14 $f(x)=\begin{cases}x, & x\in Q\\ 0, & x\notin Q\end{cases}$; $g(x)=\begin{cases}0 & x\in Q\\ x & x\notin Q\end{cases}$

then $(f - g)$ is

(A) one-one, onto (B) neither one-one, nor onto

(C) one-one but not onto (D) onto but not one-one

Q.15 If X and Y are two non-empty sets where $f : X \to Y$ is function is defined such that $f(C) = \{f(x): x \in C\}$ for $C \subseteq X$ and $f^{-1}(D) = \{x : f(x) \in D\}$ for $D \subseteq Y$ for any $A \subseteq Y$ and $B \subseteq Y$ then-

(A) $f^{-1}(f(A)) = A$

(B) $f^{-1}(f(A)) = A$ only if $f(X) = Y$

(C) $f(f^{-1}(B)) = B$ only if $B \subseteq f(x)$

(D) $f(f^{-1}(B)) = B$

Q.16 Find the range of values of t for which

$2\sin t=\frac{1-2x+5x^2}{3x^2-2x-1}$; $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(A) $\frac{3\pi}{10}\le t\le\frac{\pi}{2}$ (B) $\frac{-3\pi}{10}\le t\le\frac{\pi}{2}$

(C) $\frac{\pi}{2}\le t\le\frac{\pi}{10}$ (D) None of these

HOT Problems : Level-01

1. Find a and b if (a – 1, b + 5) = (2, 3)If A = {1,3,5}, B = {2,3}.

Ans. a = 3, b = –2

2. Find B × A Let A = {1,2}, B = {2,3,4}, C = {4,5}.

Ans. B × A = { (2,1), (2,3), (2,5), (3,1), (3,3), (3,5)}

3. If P = {1,3}, Q = {2,3,5}, find the number of relations from A to B

Ans. $2^6 = 64$

4. If A = {1,2,3,5} and B = {4,6,9}, R = {(x, y) : |x – y| is odd, x ∈A, y ∈B} Write R in roster form. Which of the following relations are functions. Give reason.

Ans. R = { (1,4), (1,6), (2,9), (3,4), (3,6), (5,4), (5,6)}

5. R = { (1,1), (2,2), (3,3), (4,4), (4,5)}

Ans. Not a function because 4 has two images.

6. R = { (2,1), (2,2), (2,3), (2,4)}

Ans. Not a function because 2 does not have a unique image.

7. Find the domain of the real function, $f(x) = \sqrt{4 - x^2}$

Ans. (–∞, –2] ∪[2, ∞)

Level-02

1. Let A = {1,2,3,4}, B = {1,4,9,16,25} and R be a relation defined from A to B as, R = {(x, y) :x ϵ A, y ϵ B and $y = x^2$}

(a) Depict this relation using arrow diagram.

(b) Find domain of R.

(c) Find range of R.

(d) Write co-domain of R.

2. Let R = { (x, y) : x, y ϵ N and y = 2x} be a relation on N. Find :

(i) Domain

(ii) Co-domain

(iii) Range

(iv) Is this relation a function from N to N

3. Find the domain and range of, f(x) = |2x – 3| – 3

4. Draw the graph of the Constant function, f : R ϵ R; f(x) = 2 x ϵ R. Also find its domain and range.

5. Let a relation be { (0,0), (2,4), (-1,2), (3,6), (1,2) } then

(i) write domain of R

(ii) write range of R

(iii) write R the set builder form

(iv) represent R by an arrow diagram

6. Find the domain and the range of the following functions

$f(x) = \frac{1}{\sqrt{5-x}}$

Level-03

1. Draw the graphs of the following real functions and hence find their range $f(x) = \frac{1}{x}$, x ϵ R and x ≠ 0

2. A function *f* is defined by $f(x) = 2x - 5$. Write down the values of

(i) $f(0)$ (ii) $f(7)$ (iii) $f(-3)$

3. Find the range of each of the following functions.

(i) $f(x) = 2 - 3x$, $x \in R$, $x > 0$.

(ii) $f(x) = x^2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.

4. Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

5. Let R be a relation from N to N defined by R = {(a, b) :a, b∈ N and $a = b^2$}. Are the following true?

(i) $(a, a) \in R$, for all $a \in N$

(ii) $(a, b) \in R$, implies $(b, a) \in R$

(iii) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.

Ans. Key - Problem for Home Practice

1-A	2-D	3-C	4-D	5-A	6-C	7-A	8-D
9-C,D	10-D	11-B	12-A	13-B	14-C	15-A	16-A
17-B	18-A	19-D	20-A	21-A	22-D	23-B	24-A
25-A							

Old IIT-JEE Questions

1-D	2-A	3-D	4-D	5-A	6-C	7-D	8-A
9-B	10-B	11-C	12-A	13-B	14-A	15-C	16-A

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☎आपका परिश्रम + हमारा मार्गदर्शन = निश्चित सफलता☎

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